

Unitarity of Quantum Theory and Closed Time-like Curves

C.J. Fewster*

*Department of Applied Mathematics and Theoretical Physics, University of Cambridge,
Silver Street, Cambridge CB3 9EW, U.K.
and Institut für theoretische Physik, Universität Bern,
Sidlerstrasse 5, CH-3012 BERN, Switzerland.[†]*

C.G. Wells[‡]

*Department of Applied Mathematics and Theoretical Physics, University of Cambridge,
Silver Street, Cambridge CB3 9EW, U.K.
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Abstract

Interacting quantum fields on spacetimes containing regions of closed time-like curves (CTCs) are subject to a non-unitary evolution X . Recently, a prescription has been proposed, which restores unitarity of the evolution by modifying the inner product on the final Hilbert space. We give a rigorous description of this proposal and note an operational problem which arises when one considers the composition of two or more non-unitary evolutions.

We propose an alternative method by which unitarity of the evolution may be regained, by extending X to a unitary evolution on a larger (possibly indefinite) inner product space. The proposal removes the ambiguity noted by Jacobson in assigning expectation values to observables localised in regions spacelike separated from the CTC region. We comment on the physical significance of the possible indefiniteness of the inner product introduced in our proposal.

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[†]Current Address

*E-mail address: fewster@butp.unibe.ch

[‡]E-mail address: C.G.Wells@damtp.cam.ac.uk

I. INTRODUCTION

Various recent studies [1–3] of perturbative interacting quantum field theory in the presence of a compact region of closed timelike curves (CTCs) have concluded that the evolution from initial states in the far past of the CTCs to final states in their far future fails to be unitary, in contrast with the situation for free fields [1,4,5]. The same conclusion has also been reached non-perturbatively for a model quantum field theory [6]. This presents many problems for the usual Hilbert space framework of quantum theory: as we describe in Section II, the Schrödinger and Heisenberg pictures are inequivalent and ambiguities arise in assigning probabilities to events occurring before [2], or spacelike separated from [7], the region of non-unitary evolution.

The main reaction to these difficulties has been to abandon the Hilbert space formulation in favour of a sum over histories approach such as the generalised quantum mechanics of Gell-Mann and Hartle (see, e.g., [8]). In particular, Hartle [9] has addressed the issue of non-unitary evolutions in generalised quantum mechanics. Nonetheless, it is of interest to see if the Hilbert space approach can be ‘repaired’ by restoring unitarity. Recently, Anderson [10] has proposed that this be done as follows. Suppose a non-unitary evolution operator X is defined on Hilbert space \mathcal{H} with inner product $\langle \cdot | \cdot \rangle$. We assume that X is bounded with bounded inverse. Anderson defines a new inner product $\langle \cdot | \cdot \rangle'$ on \mathcal{H} by $\langle \psi | \varphi \rangle' = \langle X^{-1}\psi | X^{-1}\varphi \rangle$, and denotes \mathcal{H} equipped with the new inner product as \mathcal{H}' . Regarded as a map from \mathcal{H} to \mathcal{H}' , X is clearly unitary.¹ The essence of Anderson’s proposal is to restore unitarity by regarding X in this way. Of course, one also needs to be able to represent observables as self-adjoint operators on both Hilbert spaces; Anderson has shown how this may be done by establishing a correspondence (depending on the evolution) between self-adjoint operators on \mathcal{H} and those on \mathcal{H}' . When only one non-unitary evolution is considered, this proposal is equivalent to remaining in the Hilbert space \mathcal{H} and replacing X by $U_X = (XX^*)^{-1/2}X$, i.e., the unitary part of X in the sense of the polar decomposition [11].

A curious feature of Anderson’s proposal emerges when one considers the composition of two or more consecutive periods of non-unitary evolution [12]. If an evolution Y is followed by X , one might expect that the combined evolution would be represented by the composition of the unitary parts, i.e., $U_X U_Y$. However, this does not generally agree with the unitary part of the composition, U_{XY} , and so there would be an ambiguity depending on whether one thought of the full evolution as a one-stage or two-stage journey. Anderson’s response to this is to argue that the second evolution should be treated in a different way, essentially (as we show in Section III) by replacing X by the unitary part of $X(YY^*)^{1/2}$. This removes the ambiguity mentioned above, but has the undesirable feature that the treatment of the second evolution depends on the first. In Section III, we will show that this leads to an operational problem for physicists living in a universe containing CTC regions.

It is therefore prudent to seek other means by which unitarity can be restored. In this paper, we propose a method of unitarity restoration using the mathematical technique of *unitary dilations*. This is motivated by the simple geometric observation that any linear transformation of the real line is the projection of an orthogonal transformation (called an

¹We will give a more rigorous formulation of this statement in Section III.

orthogonal dilation of the original mapping) in a larger (possibly indefinite) inner product space. To see this, note that any linear contraction on the line may be regarded as the projection of a rotation in the plane: the contraction in length along the x -axis, say, being balanced by a growth in the y -component. Similarly, a linear dilation on the line may be regarded as the projection of a Lorentz boost in two dimensional Minkowski space. This observation may be extended to operators on Hilbert spaces: it was shown by Sz.-Nagy [13] that any contraction (i.e., an operator X such that $\|X\psi\| \leq \|\psi\|$ for all ψ) has a unitary dilation acting on a larger Hilbert space. The theory was subsequently extended to non-contractive operators by Davis [14] at the cost of introducing indefinite inner product spaces. Unitary dilations have previously found physical applications in the quantum theory of open systems [15], and have also been employed by one of us in an inverse scattering construction of point-like interactions in quantum mechanics [16,17].

Put concisely, starting with a non-unitary evolution X , we pass to a unitary dilation of X , mapping between enlarged inner product spaces whose inner product may (possibly generically) be indefinite. The signature of the inner product is determined by the operator norm $\|X\|$ of X : if $\|X\| \leq 1$, the enlarged inner product spaces are Hilbert spaces, whilst for $\|X\| > 1$, they are indefinite inner product spaces (Krein spaces). Within the context of our proposal, it is therefore important to determine $\|X\|$ for any given CTC evolution operator.

Essentially, the unitary dilation proposal performs the minimal book-keeping required to restore unitarity by asserting the presence of a hidden component of the wavefunction, which is naturally associated with the CTC region. These ‘extra dimensions’ are not accessible to experiments conducted outside the CTC region, but provide somewhere for particles to hide from view, whilst maintaining global unitarity. We will see that our proposal thereby circumvents the problems associated with non-unitary evolutions mentioned above.

Of course, it is a moot point whether or not one should require a unitary evolution of quantum fields in the presence of CTCs; one might prefer a more radical approach such as that advocated by Hartle [9]. Our philosophy here is to determine the extent to which the conventional formalism of quantum theory can be repaired.

The plan of the paper is as follows. We begin in Section II by describing the implications of non-unitarity for the Hilbert space formulation of quantum mechanics and then give a rigorous description of Anderson’s proposal in Section III, where we also note the operational problem mentioned above. In Section IV, we introduce our proposal for unitarity restoration, and show how composition may be treated within this context in Section V. In Section VI, we conclude by discussing the physical significance of our proposal. There are two appendices: Appendix A contains the proof of two results required in the text, whilst Appendix B describes yet another proposal for unitarity restoration based on tensor products. However, this proposal (in contrast to that advocated by Anderson, and our own) fails to remove the ambiguity noted by Jacobson [7].

II. NON-UNITARY QUANTUM MECHANICS

As we mentioned above, a non-unitary evolution raises many problems for the standard formalism and interpretation of quantum theory, some of which we now discuss.

Firstly, the usual equivalence of the Schrödinger and Heisenberg pictures is lost. Given an evolution X of states and an observable A , we would naturally define the evolved observable A' so that for all initial states ψ , the expectation value of A' in state ψ equals the expectation of A in the evolved state $X\psi$. Explicitly, we require

$$\frac{\langle \psi | A' \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle X\psi | AX\psi \rangle}{\langle X\psi | X\psi \rangle} \quad (2.1)$$

for all ψ in the Hilbert space \mathcal{H} . If X is unitary up to a scale (i.e., $X^*X = XX^* = \lambda \mathbb{1}$, $\lambda \in \mathbb{R}^+$), then equation (2.1) is uniquely solved by the Heisenberg evolution $A' = X^{-1}AX$. On the other hand, if X is not unitary up to scale, then there is no operator A' satisfying (2.1) unless A is a scalar multiple of the identity.

For completeness, we give a proof of this fact. Defining $f(\psi)$ to equal the RHS of (2.1), and taking ψ and φ to be any orthonormal vectors, we note that linearity of A' entails

$$f(\psi) + f(\varphi) = f(\psi + \varphi) + f(\psi - \varphi), \quad (2.2)$$

whilst linearity of A implies

$$\begin{aligned} f(\psi)\|X\psi\|^2 + f(\varphi)\|X\varphi\|^2 &= \frac{1}{2} \{ f(\psi + \varphi)\|X(\psi + \varphi)\|^2 \\ &\quad + f(\psi - \varphi)\|X(\psi - \varphi)\|^2 \}. \end{aligned} \quad (2.3)$$

Multiplying φ by a phase to ensure that $\langle X\psi | X\varphi \rangle$ is imaginary (and hence that $\|X(\psi \pm \varphi)\|^2 = \|X\psi\|^2 + \|X\varphi\|^2$), we combine these relations to obtain

$$(f(\psi) - f(\varphi))(\|X\psi\|^2 - \|X\varphi\|^2) = 0, \quad (2.4)$$

which is clearly insensitive to the phase of φ and therefore holds for all orthonormal vectors ψ and φ . If X is not unitary up to scale, we choose φ and ψ so that $\|X\psi\| \neq \|X\varphi\|$. Thus $f(\psi) = f(\varphi) = F$ for some F . It follows that $f(\chi) = F$ for all $\chi \perp \text{span}\{\psi, \varphi\}$ (because $\|X\chi\|$ cannot equal both $\|X\psi\|$ and $\|X\varphi\|$) and hence for all $\chi \in \mathcal{H}$. Thus A is a scalar multiple of the identity.

Thus, the conventional equivalence of the Schrödinger and Heisenberg pictures is radically broken. If there are evolved states, there are no evolved operators, and *vice versa*. In addition, the Heisenberg picture places restrictions on the class of allowed observables. In order to preserve the canonical commutation relations, we take the evolution to be $A \rightarrow X^{-1}AX$; however, we also want to preserve self-adjointness of observables under evolution. Combining these two requirements, we conclude that A must commute with XX^* and therefore with $(XX^*)^{1/2}$ – the non-unitary part of the evolution in the sense of the polar decomposition. Thus, the claim attributed to Dirac [18] that ‘Heisenberg mechanics is the good mechanics’ carries the price of a restricted class of observables when the evolution is non-unitary.

A second problem with non-unitary evolutions, noted by Jacobson [7] (see also Hartle’s elaboration [9]) is that one cannot assign unambiguous values to expectation values of operators localised in regions spacelike separated from the CTC region. Let \mathcal{R} be a compact region spacelike separated from the CTCs, and which is contained in two spacelike slices

σ_+ and σ_- , such that σ_- passes to the past of the CTCs and σ_+ to their future. If A is an observable which is localised within \mathcal{R} , one can measure its expectation value with respect to the wavefunction on either spacelike surface. In order for these values to agree, equation (2.1) must hold with $A' = A$. If X is unitary up to scale, this is satisfied by any observable which commutes with X – in particular by all observables localised in \mathcal{R} . However, if X is not unitary up to scale, our arguments above show that there is no observable (other than multiples of the identity) for which unambiguous expectation values may be calculated. Jacobson concludes that a breakdown of unitarity implies a breakdown of causality.

Thirdly, Friedman, Papastamatiou and Simon [2] have pointed out related problems with the assignment of probabilities for events occurring before the region of CTCs. They consider a microscopic system which interacts momentarily with a measuring device before the CTC region and which is decoupled from it thereafter. The microscopic system passes through the CTC region, whilst the measuring device does not. However, the probability that a certain outcome is observed on the measuring device depends on whether it is observed before or after the microscopic system passes through the CTCs. This is at variance with the Copenhagen interpretation of quantum theory.

III. THE ANDERSON PROPOSAL

We begin by giving a rigorous description of Anderson’s proposal [10]. Let \mathcal{H} be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and suppose that the non-unitary evolution operator $X : \mathcal{H} \rightarrow \mathcal{H}$ is bounded with bounded inverse. We now define a quadratic form on \mathcal{H} by

$$q(\psi, \varphi) = \langle X^{-1}\psi | X^{-1}\varphi \rangle, \quad (3.1)$$

which (because $(X^{-1})^*X^{-1}$ is positive and X and X^{-1} are bounded) defines a positive definite inner product on \mathcal{H} whose associated norm is complete. Replacing $\langle \cdot | \cdot \rangle$ by this inner product, we obtain a new Hilbert space which we denote by \mathcal{H}' . Because \mathcal{H}' coincides with \mathcal{H} as a vector space, there is an identification mapping $\mathbf{1} : \mathcal{H} \rightarrow \mathcal{H}'$ which maps $\psi \in \mathcal{H}$ to $\psi \in \mathcal{H}'$. The inner product of \mathcal{H}' is

$$\langle \psi | \varphi \rangle' = \langle X^{-1}\mathbf{1}^{-1}\psi | X^{-1}\mathbf{1}^{-1}\varphi \rangle, \quad (3.2)$$

for $\psi, \varphi \in \mathcal{H}'$. The identification mapping is present because X^{-1} is not, strictly speaking, defined on \mathcal{H}' . As a minor abuse of notation, one can omit these mappings provided that one takes care of which inner product and adjoint are used in any manipulations. This is the approach adopted by Anderson. The advantage of writing in the identifications is that one cannot lose track of the domain or range of any operator, and adjoints automatically take care of themselves.

From equation (3.2), it is clear that $\mathbf{1}X : \mathcal{H} \rightarrow \mathcal{H}'$ (i.e., “ X regarded as a map from \mathcal{H} to \mathcal{H}' ”) is unitary – the non-unitarity of X is cancelled by that of $\mathbf{1}$. Anderson therefore adopts $\mathbf{1}X$ as the correct unitary evolution: in the Schrödinger picture, an initial state $\psi \in \mathcal{H}$ is evolved unitarily to $\mathbf{1}X\psi \in \mathcal{H}'$.

The next component in Anderson’s proposal concerns observables. Given an observable (e.g., momentum or position) represented as a self-adjoint operator A on \mathcal{H} , one needs to know how this observable is represented on \mathcal{H}' in order to evolve expectation values in the

Schrödinger picture. At first, one might imagine that A should be carried over directly using the identification mapping to form $A' = \mathbf{1}A\mathbf{1}^{-1}$. However, this idea fails because $\mathbf{1}A\mathbf{1}^{-1}$ is not self-adjoint in \mathcal{H}' unless A commutes with XX^* : an unacceptable restriction on the class of observables. Instead, Anderson proposes that A' should be defined by

$$A' = \mathbf{1}R_X A R_X^{-1} \mathbf{1}^{-1} \quad (3.3)$$

where $R_X = (XX^*)^{1/2}$ is self-adjoint and positive on \mathcal{H} . The operator $\mathbf{1}R_X$ is easily seen to be unitary, and it follows that A' is self-adjoint on \mathcal{H}' . With this definition, the expectation value of A in (normalised) state ψ evolves as

$$\langle \psi | A \psi \rangle \longrightarrow \langle \mathbf{1}X\psi | A' \mathbf{1}X\psi \rangle' = \langle U_X\psi | AU_X\psi \rangle, \quad (3.4)$$

where $U_X = R_X^{-1}X$ is the unitary part of X in the sense of the polar decomposition [11].

So far, it appears that Anderson's proposal is equivalent to Schrödinger picture evolution using U_X in the original Hilbert space, or Heisenberg evolution $A \rightarrow U_X^{-1}AU_X$. However, one must be careful with this statement when one considers the composition of two consecutive periods of evolution, say Y followed by X . We take both operators to be maps of \mathcal{H} to itself, as required by Anderson [12,22]. Proceeding naïvely, we encounter the following problem: taking the unitary parts and composing, we obtain U_XU_Y , whilst composing and taking the unitary part (i.e., considering the evolution as a whole, rather than as a two stage process) we find U_{XY} . For consistency, we would require that these evolutions should be equal up to a complex phase λ . As we show in Appendix A, this is possible if and only if X^*X commutes with YY^* and $\lambda = 1$. Composition would therefore fail in general.

In response to this, Anderson has proposed that composition be treated as follows [12]. Suppose $Y : \mathcal{H} \rightarrow \mathcal{H}$ is the first non-unitary evolution, and apply Anderson's proposal to form a Hilbert space \mathcal{H}' and an identification map $\mathbf{j} : \mathcal{H} \rightarrow \mathcal{H}'$ so that $\mathbf{j}Y$ is unitary. The next step is to form the 'push-forward' X' of the operator XR_Y to \mathcal{H}' , which is defined by

$$X' = \mathbf{j}R_Y(XR_Y)R_Y^{-1}\mathbf{j}^{-1} = \mathbf{j}R_YX\mathbf{j}^{-1}. \quad (3.5)$$

X' is decomposed as $R_{X'}U_{X'}$ in \mathcal{H}' , and $U_{X'}$ is 'pulled back' to \mathcal{H} as $\widetilde{U_{X'}} = R_Y^{-1}\mathbf{j}^{-1}U_{X'}\mathbf{j}R_Y$. Anderson states that the correct composition law is to form the product $\widetilde{U_{X'}}U_Y$. In fact, we can simplify this slightly, because

$$\widetilde{U_{X'}} = R_Y^{-1}\mathbf{j}^{-1}U_{X'}\mathbf{j}R_Y = U_{R_Y^{-1}\mathbf{j}^{-1}X'\mathbf{j}R_Y} = U_{XR_Y} \quad (3.6)$$

where we have used the fact that $U_{VXW} = VU_XW$ if V and W are unitary. Thus we can eliminate \mathcal{H}' from the discussion, and the composition rule is essentially to replace the second evolution by U_{XR_Y} rather than U_X . This is certainly consistent: for $U_{XR_Y} = U_{XY}U_Y^{-1} = U_{XY}U_Y^{-1}$, and so $U_{XR_Y}U_Y = U_{XY}$.

However, although this prescription is consistent, it has the drawback that one must know about the first non-unitary evolution in order to treat the second correctly (i.e., one must use U_{XR_Y} rather than U_X). More generally, it is easy to see that, given n consecutive evolutions X_1, \dots, X_n , one should replace each X_r by $U_{X_r R_{X_{r-1} \dots X_2 X_1}}$ for $r \geq 1$, so one needs to know about all previous evolutions at each step.

This gives rise to the following operational problem: suppose two physicists, A and B live in a universe with two isolated compact CTC regions corresponding to evolutions Y

and X respectively. Suppose that A knows about both evolutions, but B only knows about X . Thus, if A follows Anderson's proposal, she replaces these evolutions by U_Y and $U_{X R_Y}$ respectively. But B would surely replace X by U_X , which differs from $U_{X R_Y}$ unless $X^* X$ commutes with $Y Y^*$ (as a corollary of the Theorem in Appendix A). *The two physicists treat the second evolution in different ways and will therefore compute different values for expectation values of physical observables in the final state.*² This shows that, in Anderson's proposal, it is necessary to know about all non-unitary evolutions in one's past in order to treat non-unitary evolutions in one's future correctly.

For completeness, let us see how this composition law appears in the formulation of Anderson's proposal in which one modifies the Hilbert space inner product. Again we start with the evolution Y , and form the identification map $j : \mathcal{H} \rightarrow \mathcal{H}'$. In addition, we can treat the combined evolution $Z = XY$ using Anderson's proposal to form a Hilbert space \mathcal{H}'' and identification map $k : \mathcal{H} \rightarrow \mathcal{H}''$, such that kZ is unitary. The wavefunction is evolved from \mathcal{H} to \mathcal{H}' using jY , and from \mathcal{H} to \mathcal{H}'' using kZ . Thus it evolves from \mathcal{H}' to \mathcal{H}'' under $kZ(jY)^{-1} = {}_1jXj^{-1}$, where ${}_1 = kj^{-1}$ is clearly the identification mapping between \mathcal{H}' and \mathcal{H}'' . This evolution, which is forced upon us by the requirement that the wavefunction be evolved consistently, is exactly what arises from Anderson's proposal applied to the operator jXj^{-1} in \mathcal{H}' . One might expect that observables would be transformed from \mathcal{H}' to \mathcal{H}'' using the rule (3.3) applied to this evolution. However, we will now show that this is not the case.

An observable A on \mathcal{H} is represented as $A' = jR_Y A R_Y^{-1} j^{-1}$ on \mathcal{H}' , and by $A'' = kR_Z A R_Z^{-1} k^{-1}$ on \mathcal{H}'' . Thus, the transformation between A' and A'' is

$$A'' = kR_Z R_Y^{-1} j^{-1} A' j R_Y R_Z^{-1} k^{-1}. \quad (3.7)$$

Let us note that this is *not* the transformation law which follows from a naïve application of Anderson's proposal to jXj^{-1} in \mathcal{H}' , which would be of form

$$A'' = {}_1R_W A' R_W^{-1} {}_1^{-1} \quad (3.8)$$

with $W = jXj^{-1}$. Indeed, the expression (3.7) cannot generally be put into this form for any W . For suppose that there exists some W such that (3.7) and (3.8) are equivalent for all self-adjoint A' . Then $R_W = \lambda jR_Z R_Y^{-1} j^{-1}$ for some $\lambda \in \mathbb{C}$ which may be re-written as $j^{-1}R_W(j^{-1})^* = \lambda R_Z R_Y$ using the unitarity of jR_Y . The LHS is self-adjoint, so the lemma in Appendix A entails that ZZ^* and YY^* must commute, which is a non-trivial condition on X and Y when both are non-unitary. Hence in general, the transformation (3.7) is not of the form (3.8).

Thus, for consistency to be maintained, the transformation rule for observables between \mathcal{H}' and \mathcal{H}'' takes a different form from that which holds between \mathcal{H} and \mathcal{H}' or \mathcal{H}'' . We regard this as an undesirable feature of Anderson's proposal.

²One can arrange that A and B agree if A replaces Y and X by $U_{R_X Y}$ and U_X respectively, because $U_{R_X Y} = U_X^{-1} U_{XY}$. However, this would require A to know about B 's existence and ignorance of the first evolution.

IV. THE UNITARY DILATION PROPOSAL

We begin by describing the theory of unitary dilations [13,14]. Let $\mathcal{H}_1, \dots, \mathcal{H}_4$ be Hilbert spaces and let X be a bounded operator from \mathcal{H}_1 to \mathcal{H}_2 . Then an operator \hat{X} from $\mathcal{H}_1 \oplus \mathcal{H}_3$ to $\mathcal{H}_2 \oplus \mathcal{H}_4$ is called a *dilation* of X if $X = P_{\mathcal{H}_2} \hat{X}|_{\mathcal{H}_1}$ where $P_{\mathcal{H}_2}$ is the orthogonal projector onto \mathcal{H}_2 . In block matrix form, \hat{X} takes form

$$\hat{X} = \begin{pmatrix} X & P \\ Q & R \end{pmatrix}. \quad (4.1)$$

Our nomenclature follows that of Halmos [19].

Given $X : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, one may ask whether X possesses a *unitary* dilation. It turns out that such a dilation always exists, although one must pass to indefinite inner product spaces if the operator norm $\|X\|$ of X exceeds unity. One may construct a unitary dilation of X as follows. Firstly, its departure from unitarity may be quantified with the operators $M_1 = \mathbb{1} - XX^*$ and $M_2 = \mathbb{1} - X^*X$. As a consequence of the spectral theorem, we have the intertwining relations

$$X^* f(M_1) = f(M_2) X^*; \quad X f(M_2) = f(M_1) X \quad (4.2)$$

for any continuous Borel function f . The closures of the ranges of M_1 and M_2 are denoted \mathcal{M}_1 and \mathcal{M}_2 respectively.

For $i = 1, 2$, we now define $\mathcal{K}_i = \mathcal{H}_i \oplus \mathcal{M}_i$, equipped with the (possibly indefinite) inner product $[\cdot, \cdot]_{\mathcal{K}_i}$ given by

$$\left[\begin{pmatrix} \varphi \\ \Phi \end{pmatrix}, \begin{pmatrix} \psi \\ \Psi \end{pmatrix} \right]_{\mathcal{K}_i} = \langle \varphi | \psi \rangle + \langle \Phi | \operatorname{sgn} M_i \Psi \rangle, \quad (4.3)$$

where the inner products on the right are taken in \mathcal{H} and $\operatorname{sgn} M_i = |M_i|^{-1} M_i$ where $|M_i| = (M_i^* M_i)^{1/2}$. It is easy to show that $\operatorname{sgn} M_i$ is positive if $\|X\| \leq 1$, in which case $[\cdot, \cdot]_{\mathcal{K}_i}$ is positive definite; however, for $\|X\| > 1$, the inner products above are indefinite, and \mathcal{K}_1 and \mathcal{K}_2 are *Krein spaces* (for details on the theory of operators in indefinite inner product spaces, see the monographs [20,21]). It is important to remember that the \mathcal{K}_i also have a positive definite inner product from their original definition as a direct sum of Hilbert spaces.³ Thus a bounded linear operator A from \mathcal{K}_1 to \mathcal{K}_2 has two adjoints: the Hilbert space adjoint A^* , and the Krein space adjoint, which we denote A^\dagger . It is a simple exercise to show that A^\dagger is given by

$$A^\dagger = J_1 A^* J_2, \quad (4.4)$$

where the operators J_i defined on \mathcal{K}_i are unitary involutions given by $J_i = \mathbb{1}_{\mathcal{H}_i} \oplus \operatorname{sgn}(M_i)$.

Next, we define a dilation $\hat{X} : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ of X by

$$\hat{X} = \begin{pmatrix} X & -\operatorname{sgn}(M_1) |M_1|^{1/2} \\ |M_2|^{1/2} & X^* |M_1| \end{pmatrix}, \quad (4.5)$$

³In fact, this inner product determines the topology of \mathcal{K}_i .

which has adjoint \hat{X}^\dagger given by (4.4) as

$$\hat{X}^\dagger = \begin{pmatrix} X^* & \text{sgn}(M_2)|M_2|^{1/2} \\ -|M_1|^{1/2} & \text{sgn}(M_1)X|_{\mathcal{M}_2}\text{sgn}(M_2) \end{pmatrix}. \quad (4.6)$$

It is then a matter of computation using the intertwining relations to show that $\hat{X}^\dagger \hat{X} = \mathbb{1}_{\mathcal{K}_1}$ and $\hat{X} \hat{X}^\dagger = \mathbb{1}_{\mathcal{K}_2}$. \hat{X} is therefore a unitary dilation of X .

The construction we have given is not unique. For suppose that \mathcal{N}_1 and \mathcal{N}_2 are Krein spaces, and that $U_i : \mathcal{M}_i \rightarrow \mathcal{N}_i$ are unitary (with respect to the indefinite inner products). Then

$$\tilde{X} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U_2 \end{pmatrix} \hat{X} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U_1^\dagger \end{pmatrix} \quad (4.7)$$

is also a unitary dilation of X , mapping between $\mathcal{H} \oplus \mathcal{N}_1$ and $\mathcal{H} \oplus \mathcal{N}_2$. Because this just amounts to a redefinition of the auxiliary spaces, it carries no additional physical significance. One may show that all other unitary dilations of X require the addition of larger auxiliary spaces than the \mathcal{M}_i (for example, one could dilate \hat{X} further). Thus \hat{X} is the minimal unitary dilation of X up to unitary equivalence of the above form.

Having described the general theory, let us now apply it to the case of interest. For simplicity, we assume that the Hilbert spaces of initial and final states are identical, so $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. We also assume that the evolution operator X is bounded with bounded inverse. If the initial hypersurface contains regions which are causally separate from the CTC region, we assume that X has been normalised to be unitary on states localised in such regions. We point out that such exterior regions may not exist – even if the CTC region is itself compact. Consider, for example, a spacetime that is asymptotically (the universal cover of) anti-de Sitter space. In such a spacetime, hypersurfaces sufficiently far to the future and far to the past of the CTC region will be entirely contained within the CTC region’s light cone and there will be no exterior region on which to set up our normalisation. We may normalise the evolution operator on hypersurfaces for which an exterior region may be identified and extend arbitrarily to those surfaces where no such region exists. Indeed, it is entirely possible that every point in spacetime is contained in the light cone of the CTC region; in this case we give up any attempt to find a ‘physical’ normalisation for the evolution operator.

The spaces \mathcal{M}_1 and \mathcal{M}_2 are defined as above. Note that we have the polar decomposition $X = (XX^*)^{1/2}U$, where U is a *unitary* operator because X is invertible. As a consequence of the intertwining relations, we have

$$UM_2 = M_1U \quad (4.8)$$

and hence that $\mathcal{M}_1 = U\mathcal{M}_2$. Thus the \mathcal{M}_i are isomorphic as Hilbert spaces. Moreover, U is also unitary with respect to the indefinite inner products on the auxiliary spaces \mathcal{M}_1 and \mathcal{M}_2 , which follows from the identity $U\text{sgn}(M_2) = \text{sgn}(M_1)U$. We can therefore use the freedom provided by equation (4.7) to arrange that the same auxiliary space is used both before and after the evolution.

Our proposal is the following. Given a non-unitary evolution X , there exists an (indefinite) auxiliary space \mathcal{M} (isomorphic to the \mathcal{M}_i) and a unitary dilation $\tilde{X} : \mathcal{K} \rightarrow \mathcal{K}$ of

X , where $\mathcal{K} = \mathcal{H} \oplus \mathcal{M}$. We regard this as describing the full physics of the situation: on \mathcal{K} , the evolution is unitary, whilst its restriction to the original Hilbert space \mathcal{H} yields the non-unitary operator X . The auxiliary space \mathcal{M} represents degrees of freedom localised within the CTC region, not directly accessible to experiments outside.⁴

Observables are defined as follows. Given any self-adjoint operator A on \mathcal{H} , we define the corresponding observable on \mathcal{K} :

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.9)$$

The form of \tilde{A} is chosen to prevent the internal degrees of freedom being probed from outside.

Let us point out that many features of this proposal can only be determined in the context of a particular evolution X and therefore a particular CTC spacetime. There are, however, various model independent features of our proposal, which we discuss below.

Predictability Because the initial state involves degrees of freedom not present on the initial hypersurface (i.e., the component of the wavefunction in \mathcal{M}), it is clear that – as far as physical measurements are concerned – there is some loss of predictability in the final state. This problem can be circumvented by the requirement that the initial state should have no component in \mathcal{M} .

Expectation Values Let us examine the evolution of the expectation value of \tilde{A} . On the premise that the initial state has no component in \mathcal{M} and takes the vector form $(\psi, 0)^T$, the initial expectation value of \tilde{A} is

$$\frac{\left[\begin{pmatrix} \psi \\ 0 \end{pmatrix}, \tilde{A} \begin{pmatrix} \psi \\ 0 \end{pmatrix} \right]_{\mathcal{K}_2}}{\left[\begin{pmatrix} \psi \\ 0 \end{pmatrix}, \begin{pmatrix} \psi \\ 0 \end{pmatrix} \right]_{\mathcal{K}_2}} = \frac{\langle \psi | A \psi \rangle}{\langle \psi | \psi \rangle}, \quad (4.10)$$

i.e., the expectation value of A in state ψ . After evolution, the expectation value is

$$\frac{\left[\tilde{X} \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \tilde{A} \tilde{X} \begin{pmatrix} \psi \\ 0 \end{pmatrix} \right]_{\mathcal{K}_2}}{\left[\tilde{X} \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \tilde{X} \begin{pmatrix} \psi \\ 0 \end{pmatrix} \right]_{\mathcal{K}_2}} = \frac{\langle X \psi | A X \psi \rangle}{\langle \psi | \psi \rangle}. \quad (4.11)$$

It is important to note that both denominators are equal to $\|\psi\|^2$ (because the full evolution is unitary) – this removes many of the problems encountered in Section II.

In particular, let us return to the problem noted by Jacobson [7], writing \mathcal{R} for the region spacelike separated from the CTC region, and taking X to be the evolution from states on σ_- to states on σ_+ . We assume (as in [7]) that X acts as the identity on $\mathcal{H}_{\mathcal{R}}$, the subspace of states supported in \mathcal{R} . Any local observable associated with \mathcal{R} should vanish on the orthogonal complement of $\mathcal{H}_{\mathcal{R}}$ in \mathcal{H} : accordingly, it follows that $X^* A X = A$, and hence

⁴*Indirectly*, we can infer their presence by analysing X .

that the expectation value is independent of the choice of hypersurface (σ_+ or σ_-) on which it is computed. Thus Jacobson's ambiguity is avoided for all local observables associated with regions spacelike separated from the causality-violating region. More generally, it is avoided for all observables A such that $A = X^*AX$. This is satisfied if the range of A is contained in $\mathcal{U} = \ker M_1 \cap \ker M_2 \subset \mathcal{H}$ and A commutes with the restriction $X|_{\mathcal{U}}$ of X to \mathcal{U} .

In addition, the breakdown of the Copenhagen interpretation noted in [2] is avoided as a direct consequence of the unitarity of \tilde{X} .

Time Reversal Let us suppose the existence of an anti-unitary involution T on \mathcal{H} implementing time reversal. The *time reverse* X_{rev} of X is given by $X_{\text{rev}} = TXT^{-1}$; X is said to be *time reversible* if $X_{\text{rev}} = X^{-1}$. We would like to understand how the time reversal properties of \hat{X} are related to those of X . For convenience we will work in terms of \hat{X} ; the discussion may be rephrased in terms of \tilde{X} by inserting suitable unitary operators between the \mathcal{M}_i and \mathcal{M} .

First, we must define the time reversal of \hat{X} . The natural definition is

$$(\hat{X})_{\text{rev}} = \begin{pmatrix} T & 0 \\ 0 & T|_{\mathcal{M}_2} \end{pmatrix} \hat{X} \begin{pmatrix} T^{-1} & 0 \\ 0 & (T|_{\mathcal{M}_1})^{-1} \end{pmatrix}, \quad (4.12)$$

which entails that time reversal and dilation commute in the sense that $(\hat{X})_{\text{rev}} = \widehat{X_{\text{rev}}}$. However, because dilation and inversion do not commute (i.e., $(\hat{X})^{-1} \neq \widehat{X^{-1}}$) unless X is unitary, we find that a time reversible evolution X does not generally yield a time reversible dilation:

$$(\hat{X})_{\text{rev}} = \widehat{X_{\text{rev}}} = \widehat{X^{-1}} \neq (\hat{X})^{-1}. \quad (4.13)$$

Thus if X is non-unitary and time reversible, then \hat{X} is not time reversible. On the other hand, suppose that \hat{X} is time reversible. Then $\widehat{X_{\text{rev}}} = \widehat{X^*}$ from which it follows that X would obey the modified reversal property $X_{\text{rev}} = X^*$. It would be interesting to determine, for concrete CTC models, whether X obeys the usual time reversal property $X_{\text{rev}} = X^{-1}$ or the modified property $X_{\text{rev}} = X^*$ (of course it might not obey either property).

To summarise this section, we have seen how unitarity can be restored using the method of unitary dilations, thereby removing the problems associated with non-unitary evolutions. Any observable on \mathcal{H} defines an observable in our proposal.

V. COMPOSITION

We have described how a single non-unitary evolution may be dilated to a unitary evolution between enlarged inner product spaces. In what sense does our proposal respect the composition of two (or more) non-unitary evolutions?

Let us consider two evolutions X and Y on \mathcal{H} and their composition XY . We define the M_i and \mathcal{M}_i as before and introduce $N_1 = \mathbb{1} - YY^*$, $N_2 = \mathbb{1} - Y^*Y$ and $\mathcal{N}_i = \overline{\text{Ran } N_i}$ to be the closure of the range of N_i for $i = 1, 2$. As before, we can construct dilations \hat{X} and \hat{Y} . However, because $\hat{X} : \mathcal{H} \oplus \mathcal{M}_1 \rightarrow \mathcal{H} \oplus \mathcal{M}_2$ and $\hat{Y} : \mathcal{H} \oplus \mathcal{N}_1 \rightarrow \mathcal{H} \oplus \mathcal{N}_2$, it is not immediately apparent how the dilations may be composed. The solution is to dilate both \hat{X} and \hat{Y} further, as follows: $\check{Y} : \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_1 \rightarrow \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_2$ is given by

$$\check{Y} = \begin{pmatrix} Y & 0 & -\operatorname{sgn} N_1 |N_1|^{1/2} \\ 0 & \mathbb{1}_{\mathcal{M}_1} & 0 \\ |N_2|^{1/2} & 0 & Y^*|_{\mathcal{N}_1} \end{pmatrix}, \quad (5.1)$$

and $\check{X} : \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_2 \rightarrow \mathcal{H} \oplus \mathcal{M}_2 \oplus \mathcal{N}_2$ is given by

$$\check{X} = \begin{pmatrix} X & -\operatorname{sgn} M_1 |M_1|^{1/2} & 0 \\ |M_2|^{1/2} & X^*|_{\mathcal{M}_1} & 0 \\ 0 & 0 & \mathbb{1}_{\mathcal{N}_2} \end{pmatrix}. \quad (5.2)$$

The product $\check{X}\check{Y}$ is given by

$$\check{X}\check{Y} = \begin{pmatrix} XY & -\operatorname{sgn} M_1 |M_1|^{1/2} & -X \operatorname{sgn} N_1 |N_1|^{1/2} \\ |M_2|^{1/2} Y & X^*|_{\mathcal{M}_1} & -|M_2|^{1/2} \operatorname{sgn} N_1 |N_1|^{1/2} \\ |N_2|^{1/2} & 0 & Y^*|_{\mathcal{N}_1} \end{pmatrix}, \quad (5.3)$$

and is a unitary dilation of XY , mapping from $\mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_1$ to $\mathcal{H} \oplus \mathcal{M}_2 \oplus \mathcal{N}_2$.

This state of affairs is quite natural: we have argued that each CTC region carries with it its own auxiliary space (isomorphic to the \mathcal{M}_i and the \mathcal{N}_i); one would therefore expect that the combined evolution should be associated with the direct sum of these auxiliary spaces. However, in order to show how our proposal respects composition, we need to show how the product $\check{X}\check{Y}$ is related to the dilation \widehat{XY} arising from the prescription (4.5). To this end, we introduce $P_1 = \mathbb{1} - XY Y^* X^*$, $P_2 = \mathbb{1} - Y^* X^* X Y$ and $\mathcal{P}_i = \overline{\operatorname{Ran} P_i}$. Note that

$$P_1 = M_1 + X N_1 X^* \quad \text{and} \quad P_2 = N_2 + Y^* M_2 Y. \quad (5.4)$$

Now let

$$Q_1 = \begin{pmatrix} |M_1|^{1/2} \\ |N_1|^{1/2} X \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} |M_2|^{1/2} Y \\ |N_2|^{1/2} \end{pmatrix}, \quad (5.5)$$

and define U_i ($i = 1, 2$) on $\operatorname{Ran} |P_i|^{1/2} \subset \mathcal{P}_i$ by $U_i = Q_i |P_i|^{-1/2}$. The U_i are easily seen to be isometries (with respect to the appropriate inner products) from their domains into $\mathcal{M}_i \oplus \mathcal{N}_i$ such that $Q_i|_{\overline{\operatorname{Ran} P_i}} = U_i |P_i|^{1/2}$. Provided that $\mathcal{Q}_i = \overline{Q_i \operatorname{Ran} P_i}$ is orthocomplemented in $\mathcal{M}_i \oplus \mathcal{N}_i$ (in the indefinite inner product), one may then show that

$$P_{\mathcal{H} \oplus \mathcal{Q}_2} \check{X}\check{Y}|_{\mathcal{H} \oplus \mathcal{Q}_1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} XY & -\operatorname{sgn} P_1 |P_1|^{1/2} \\ |P_2|^{1/2} & (XY)^*|_{\mathcal{P}_1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U_1^\dagger \end{pmatrix}, \quad (5.6)$$

where $P_{\mathcal{H} \oplus \mathcal{Q}_2}$ is the orthoprojector onto $\mathcal{H} \oplus \mathcal{Q}_2$. Thus $\check{X}\check{Y}$ is a dilation of an operator isometrically equivalent to \widehat{XY} . The isometries act non-trivially only on the auxiliary spaces and have no physical significance. The extra dimensions introduced by the dilation are also to be expected because the combined evolution $Z = XY$ may be factorised in many different ways; hence the two individual evolutions carry more information than their combination.

The assumption that the \mathcal{Q}_i are orthocomplemented is easily verified if the operators U_i are bounded, for in this case, they may be extended to unitary operators on the whole of \mathcal{P}_i . Then \mathcal{Q}_i is the unitary image of a Krein space and is orthocomplemented by Theorem VI.3.8 in [20]. U_1 is bounded if there exists K such that $\|P_1 \psi\| < \epsilon$ only if $\|M_1 \psi\| + \|N_1 X \psi\| < K \epsilon$

for all sufficiently small $\epsilon > 0$. Similarly, U_2 is bounded if $\|P_1\psi\| < \epsilon$ only if $\|M_2Y\psi\| + \|N_2\psi\| < K\epsilon$ for all sufficiently small $\epsilon > 0$. Physically, this equates to the reasonable condition that the combined evolution can be ‘almost unitary’ on a given state only if the individual evolutions are also ‘almost unitary’.

As a particular instance of the above, we consider the case where Y is unitary. The N_i therefore vanish and the \mathcal{N}_i are trivial; in addition, $P_1 = M_1$ and $P_2 = Y^*M_2Y$. The operator \check{Y} is

$$\check{Y} = \begin{pmatrix} Y & 0 \\ 0 & \mathbb{1}_{\mathcal{M}_1} \end{pmatrix} \quad (5.7)$$

and $\check{X} = \hat{X}$. The combined evolution is thus

$$\check{X}\check{Y} = \hat{X} \begin{pmatrix} Y & 0 \\ 0 & \mathbb{1}_{\mathcal{M}_1} \end{pmatrix} \quad (5.8)$$

which is unitarily equivalent to \widehat{XY} in the sense that

$$\hat{X} \begin{pmatrix} Y & 0 \\ 0 & \mathbb{1}_{\mathcal{M}_1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & Y \end{pmatrix} \widehat{XY}. \quad (5.9)$$

We emphasise that the first factor on the RHS has no physical significance and is merely concerned with mapping the auxiliary spaces \mathcal{P}_2 to \mathcal{M}_2 in a natural way.

To conclude this section, we make three comments. Firstly, note that if A belongs to the class of observables which avoid the Jacobson ambiguity for each CTC region individually, then it also avoids this ambiguity for the combined evolution; for if $A = X^*AX = Y^*AY$, then certainly $A = Y^*X^*AXY$. Thus there is no ‘multiple Jacobson ambiguity’. Secondly, in our proposal one does not need to know the past history of the universe in order to evolve forward from any given time, because the auxiliary degrees of freedom associated with one CTC region are essentially passive ‘spectators’ during the evolution associated with any other such region. This is in contrast with the composition rule proposed by Anderson [12]. Thirdly, one might ask [22] what would happen if the non-unitary evolution was continuous rather than occurring in discrete steps. This question could be tackled using a suitable generalisation of the theory of unitary dilations of semi-groups discussed by Davies [15].

VI. CONCLUSION

We have examined Anderson’s proposal [10] for restoring unitarity to quantum evolution in CTC spacetimes, and noted an operational problem arising when one considers the composition of two or more non-unitary evolutions. Instead, we have advocated a new method for the restoration of unitarity, based on the mathematical theory of unitary dilations, which does respect composition under certain reasonable conditions. Because unitarity is restored on the full inner product space, problems associated with non-unitary evolutions such as Jacobson’s ambiguity are avoided.

Our philosophy here has been to regard the non-unitarity of X as a signal that the full physics (and a unitary evolution) is being played out on a larger state space than is observed.

This bears some resemblance to the situation in special relativity, where time dilation signals that one must pass to spacetime (and an indefinite metric) in order to restore an orthogonal transformation between reference frames. (Indeed, the Lorentz boost in two dimensional Minkowski space is precisely an orthogonal dilation of the time dilation effect).

For our case of interest, the physical picture is that the auxiliary space \mathcal{M} corresponds to degrees of freedom within the CTC region. Non-unitarity of the evolution signals that a particle cannot pass through the CTC region unscathed: part of the initial state becomes trapped in the auxiliary space corresponding to the CTCs. A similar conclusion is espoused by three of the authors of [23].

In the case in which X has norm less than or equal to unity (so that the full space \mathcal{K} has a positive definite inner product), this effect has a relatively simple interpretation. Namely, there is a non-zero probability that an incident particle will never emerge from the CTC region. To see how this can occur, we note that computations of the propagator (see particularly [6]) proceed by requiring consistency of the evolution round the CTCs. We suggest that part of the incident state becomes trapped in order to achieve this consistency.

On the other hand, perturbative calculations in $\lambda\phi^4$ theory by Boulware [1] suggest that $\|X\|$ could well exceed unity. In this case, \mathcal{K} is an indefinite Krein space, and it would apparently be possible that the ‘probability’ of the particle escaping from the CTC region could be greater than one. In principle, one might try to avoid this by seeking natural positive definite subspaces of the initial and final Krein spaces. The obvious choice would be to take the initial Hilbert space to be \mathcal{H} and the final Hilbert space to be the image of \mathcal{H} under \tilde{X} . However, this may lead to some problems in defining observables on the final Hilbert space. If one decides to face the problem directly (which seems preferable), one would be forced to conclude that CTCs are incompatible with the twin requirements of unitarity *and* a Hilbert space structure. The initial and final state spaces would naturally be Krein spaces. This would not be entirely unexpected: studies of quantum mechanics on the ‘spinning cone’ spacetime [24] have concluded that the inner product becomes indefinite precisely inside the region of CTCs. ‘Probabilities’ greater than unity would denote the breakdown of the theory in a manner analogous to the Klein paradox (see the extensive discussion in the monograph of Fulling [25]) in which strong electrostatic fields force the Klein-Gordon inner product to be indefinite. In our case, it is the geometry of spacetime which leads us to an indefinite inner product. We expect that particle creation would occur in this case, as it does in the usual Klein paradox.

The Klein paradox can be resolved by treating the electromagnetic field as a dynamic field, rather than as a fixed external field. Particles are created in a burst as the field collapses (unless it is maintained by some external agency). In our case it seems reasonable that, in the context of a full quantum theory of gravity, a burst of particle creation occurs and the CTC region collapses. This is essentially the content of Hawking’s Chronology Protection Conjecture [26]. Thus the emergence of Krein spaces in our proposal may be interpreted as a signal for the instability of the CTC spacetime.

Finally, our treatment has been entirely in terms of states and operators; it would be interesting to see how it translates into density matrices and the language of generalised quantum mechanics [8].

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APPENDIX A:

In this appendix, we prove the following

Theorem *Suppose X and Y are bounded with bounded inverses. Then $U_{XY} = \lambda U_X U_Y$ if and only if X^*X commutes with YY^* and $\lambda = 1$.*

Proof: Starting with the sufficiency, we note that $Z = (X^*)^{-1}(X^*X)^{1/2}(YY^*)^{-1/2}X^{-1}$ is positive and squares to give $(XY Y^* X^*)^{-1}$ (using the commutation property). It follows that Z is equal to the unique positive square root of $(XY Y^* X^*)^{-1}$ and hence that

$$U_{XY} = (XY Y^* X^*)^{-1/2}XY = (X^*)^{-1}(X^*X)^{1/2}(YY^*)^{-1/2}Y. \quad (\text{A1})$$

Using the fact that $(X^*)^{-1}(X^*X)^{1/2} = U_X$, we have proved sufficiency.

To demonstrate necessity, we note that $U_{XY} = \lambda U_X U_Y$ only if

$$X^*(XY Y^* X^*)^{-1/2}X = \lambda(X^*X)^{1/2}(YY^*)^{-1/2}. \quad (\text{A2})$$

It follows that the RHS must be self-adjoint and positive. We now apply the following Lemma:

Lemma *Suppose that A and B are bounded with bounded inverses and self-adjoint, and suppose that αAB is self-adjoint and positive for some $\alpha \in \mathbb{C}$, $\alpha \neq 0$. Then $\alpha = \pm 1$ and A and B commute.*

Proof: Because αAB is self-adjoint, we have

$$\alpha AB = \alpha^* BA. \quad (\text{A3})$$

Now note that

$$\begin{aligned} \alpha^*(\alpha AB - z)^{-1} &= \alpha^*(\alpha^* BA - z)^{-1} \\ &= \alpha B(\alpha AB - z\alpha^*/\alpha)^{-1}B^{-1}. \end{aligned} \quad (\text{A4})$$

Because AB has non-empty spectrum on the positive real axis and because the resolvent $(\alpha AB - z)^{-1}$ is an analytic operator valued function of z in $\mathbb{C} \setminus \mathbb{R}^+$, we conclude that α^*/α must be real and positive. Accordingly, $\alpha = \pm 1$ and equation (A3) implies that A and B commute. \square

In our case, this implies that $\lambda = \pm 1$ and that X^*X commutes with YY^* . Moreover, because the two square roots on the RHS of equation (A2) are positive and commute, we conclude that $\lambda = 1$ in order that the RHS be positive. \square

APPENDIX B:

Here, we consider another possible method for the restoration of unitarity which, however, suffers from problems related to Jacobson's ambiguity. Instead of focussing on direct sums of Hilbert spaces, this proposal uses tensor products and always maintains a positive definite inner product. We start with $X : \mathcal{H} \rightarrow \mathcal{H}$, bounded with bounded inverse and non-unitary as before, and define a new Hilbert space $\mathcal{H}_X = (\mathbb{1} \otimes X)\Sigma$, where $\Sigma \subset \mathcal{H} \otimes \mathcal{H}$ is the closure of the space of finite linear combinations of terms of form $\psi \otimes \psi$ for $\psi \in \mathcal{H}$. Similarly, we define $\mathcal{H}_{X^{-1}} = (\mathbb{1} \otimes X^{-1})\Sigma$. Now define the operator $\tilde{X} = X \otimes X^{-1}$ restricted to \mathcal{H}_X . Clearly, $\tilde{X}(\psi \otimes X\psi) = \varphi \otimes X^{-1}\varphi$ where $\varphi = X\psi$, and so $\tilde{X} : \mathcal{H}_X \rightarrow \mathcal{H}_{X^{-1}}$. Moreover,

$$\begin{aligned} \langle \tilde{X}(\psi \otimes X\psi) | \tilde{X}(\varphi \otimes X\varphi) \rangle &= \langle X\psi \otimes \psi | X\varphi \otimes \varphi \rangle \\ &= \langle X\psi | X\varphi \rangle \langle \psi | \varphi \rangle \\ &= \langle \psi \otimes X\psi | \varphi \otimes X\varphi \rangle \end{aligned} \tag{B1}$$

and therefore \tilde{X} is a unitary operator from \mathcal{H}_X to $\mathcal{H}_{X^{-1}}$.

Let us examine the structure of this proposal in more detail. First, there is a natural transposition operation \mathcal{T} on $\mathcal{H} \otimes \mathcal{H}$: $\mathcal{T}(\varphi \otimes \psi) = \psi \otimes \varphi$. It is easy to see that \tilde{X} is the restriction of \mathcal{T} to \mathcal{H}_X : hence all the information about X is encoded into the definition of \mathcal{H}_X . Have we lost any information in this process? Suppose $\mathcal{H}_X = \mathcal{H}_Y$ for two distinct operators X and Y . Then $\mathbb{1} \otimes Z$ is a bounded invertible linear map (though not necessarily unitary) of Σ onto itself, where $Z = X^{-1}Y$. Because \mathcal{T} restricts to the identity on Σ , we require $\psi \otimes Z\psi = Z\psi \otimes \psi$ for each $\psi \in \mathcal{H}$. Taking an inner product with $\phi \otimes \psi$ for some ϕ , we obtain

$$\langle \phi | \psi \rangle \langle \psi | Z\psi \rangle = \langle \phi | Z\psi \rangle \langle \psi | \psi \rangle. \tag{B2}$$

Because ϕ is arbitrary, ψ is therefore an eigenvector of Z . But ψ was also arbitrary and therefore $Z = \lambda \mathbb{1}$ for some constant $\lambda \in \mathbb{C} \setminus \{0\}$. Thus $Y = \lambda X$, so this construction loses exactly one scalar degree of freedom. Effectively, we have lost the (scalar) operator norm $\|X\|$ of X , but no other information.

We have therefore restored unitarity at the price of introducing a second Hilbert space and correlations between the two. The evolution on the large space is unitary. This fits well with the picture of acausal interaction between the initial space and the CTC region in its future. The physical interpretation is as follows: the 'time machine' contains a copy of the external universe, which evolves backwards in time, starting with the final state of the quantum fields and ending with their initial state. It is impossible to prepare the initial state of the CTC region independently from the initial state of the exterior quantum fields.

However, problems arise when observables are defined. Here, observables on the initial space are naturally defined to be self-adjoint operators on $\mathcal{H} \otimes \mathcal{H}$ with \mathcal{H}_X as an invariant subspace (observables on the final space would have $\mathcal{H}_{X^{-1}}$ invariant). An operator of form $A \otimes B$ maps \mathcal{H}_X to itself only if $B = XAX^{-1}$; combining this with the requirement of self-adjointness, one finds that A must commute with X^*X and its powers. Thus this proposal places restrictions on the class of allowed observables.

The requirement that \mathcal{H}_X be an invariant subspace for all observables was adopted so that our space of initial states is invariant under the unitary groups generated by observables

(e.g. translations). If we relax this, and define observables to be self-adjoint operators on $\mathcal{H} \otimes \mathcal{H}$, it appears that $A \otimes \mathbb{1}$ corresponds naturally to the operator A on \mathcal{H} . However, this suffers from the ambiguity pointed out by Jacobson [7].

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